# Self-Similar Solutions with Elliptic Symmetry for the Compressible Euler and Navier-Stokes Equations in $R^N$

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**Abstract**

Based on Makino's solutions with radially symmetry, we extend the corresponding ones with elliptic symmetry for the compressible Euler and Navier-Stokes equations in $R^N$ $(N \geq 2)$. By the separation method, we reduce the Euler and Navier-Stokes equations into $1+N$ differential functional equations. In detail, the velocity is constructed by the novel Emden dynamical system:

$$\begin{cases} \ddot{a}_i(t) = \dfrac{\xi}{a_i(t)\left(\prod\limits_{k=1}^{N} a_k(t)\right)^{\gamma-1}}, \text{ for } i = 1, 2, ...., N \\ a_i(0) = a_{i0} > 0, \ \dot{a}_i(0) = a_{i1} \end{cases} \tag{1}$$

with arbitrary constants $\xi$, $a_{i0}$ and $a_{i1}$. Some blowup phenomena or global existences of the solutions obtained could be shown.

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## 1 Introduction

The compressible Euler or Navier-Stokes equations are written as the follows:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0 \\ \rho \left[\vec{u}_t + (\vec{u} \cdot \nabla) \vec{u}\right] + K \rho^{\gamma} = \mu \Delta \vec{u} \end{cases} \tag{2}$$

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where the density $\rho=\rho(t,\vec{x})$ and velocity $\vec{u}=\vec{u}(t,\vec{x})=(u_1,u_2,....,u_N)\in R^N$ with $\vec{x}=(x_1,x_2,...,x_N)\in R^N$. And $K>0$, $\gamma\geq 1$ and $\mu\geq 0$ are constants. If $\mu=0$, the system (2) is the Euler equations; if $\mu>0$, the system (2) is the Navier-Stokes equations.

The Euler and Navier-Stokes equations (2) are the very fundamental models in fluid mechanics [2] and [1]. Searching particular solutions for the systems are the important part in mathematical physics for understanding their nonlinear phenomena. By the separation method, Makino firstly obtained the radial symmetry solutions for the Euler or Navier-Stokes equations (2) in 1993 [4]. After that there are some other ways to construct some particular solutions [3] and [8] for these systems.

It is natural to seek solutions with elliptic symmetry for the Euler or Navier-Stokes equations (2), based on the previous work. In this brief article, we could generalize Makino's solutions to the corresponding ones with elliptical symmetry and drift phenomena for these systems in the following theorem:

**Theorem 1** *To the Euler and Navier-Stokes equations (2) in $R^N$, there exists a family of solutions:*

$$\begin{cases} \rho=\dfrac{f(s)}{\prod\limits_{k=1}^{N}a_k} \\ u_i=\frac{\dot{a}_i}{a_i}\left(x_i+d_i\right)\ \text{for } i=1,2,....,N \end{cases} \tag{3}$$

*where*

$$f(s)=\begin{cases} \alpha e^{-\frac{\xi}{2K}s} & \text{for } \gamma=1 \\ \max\left(\left(-\frac{\xi(\gamma-1)}{2K\gamma}s+\alpha\right)^{\frac{1}{\gamma-1}},\ 0\right) & \text{for } \gamma>1 \end{cases} \tag{4}$$

*with $s=\sum\limits_{k=1}^{N}\frac{(x_k+d_k)^2}{a_k(t)^2}$, arbitrary constants $\alpha\geq 0$, $d_k$ and $\xi$;*
*and the auxiliary functions $a_i=a_i(t)$ satisfy the Emden dynamical system:*

$$\begin{cases} \ddot{a}_i=\dfrac{\xi}{a_i\left(\prod\limits_{k=1}^{N}a_k\right)^{\gamma-1}},\ \text{for } i=1,2,....,N \\ a_i(0)=a_{i0}>0,\ \dot{a}_i(0)=a_{i1} \end{cases} \tag{5}$$

*with arbitrary constants $a_{i0}$ and $a_{i1}$.*
*In particular, with $\gamma=1$,*
*(1a) for $\xi<0$, the solutions (3) blow up on a finite time;*
*(1b) for $\xi>0$, the solutions (3) exists globally.*
*with $\gamma>1$,*
*(2a) for $\xi<0$ and some $a_{i1}<0$, the solutions (3) blow up on or before the finite time*

$$T=\min(-a_{i0}/a_{i1}:a_{1i}<0,i=1,2,...,N); \tag{6}$$

*(2b) for $\xi>0$ and $a_{i1}\geq 0$ the solutions (3) exist globally.*

**Remark 2** *When $a_1 = a_2 = .... = a_N = a(t)$, the solutions are with radial symmetry and the Emden dynamical system (5) returns to the conventional Emden equation:*

$$\begin{cases} \ddot{a}(t) = \frac{\xi}{a(t)^{N(\gamma-1)+1}} \\ a(0) = a_0 > 0,\ \dot{a}(0) = a_1. \end{cases} \tag{7}$$

*This class of analytical solutions (3) with radial symmetry for the compressible Euler equations (2) was first discovered by Makino in [4]. Otherwise, the solutions (3) are with elliptical symmetry for $N \geq 2$.*

# 2 The Separation Method

Very recently, Yeung and Yuen in [5] discovered the implicit or explicit functions for the mass equations $(3)_1$. In this section, we apply their result in the explicit expression to have the following lemma:

**Lemma 3 (Lemma 1 in [5])** *For the equation of conservation of mass:*

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0, \tag{8}$$

*there exist solutions,*

$$\begin{cases} \rho = \frac{f\left(\frac{x_1+d_1}{a_1(t)}, \frac{x_2+d_2}{a_2(t)}, ...., \frac{x_N+d_N}{a_N(t)}\right)}{\prod\limits_{i=1}^{N} a_i(t)} \\ u_i = \frac{\dot{a}_i(t)}{a_i(t)}(x_i + d_i) \ \textit{for } i = 1, 2, ...., N \end{cases} \tag{9}$$

*with an arbitrary $C^1$ function $f \geq 0$ and $a_i(t) > 0$ and constants $d_i$.*

For better understanding the lemma, the proof is also provided here.

**Proof.** We plug the functions

$$\begin{cases} \rho = \rho(t, \vec{x}) \\ u_i = \frac{\dot{a}_i(t)}{a_i(t)}(x_i + d_i) \ \text{for } i = 1, 2, ...., N, \end{cases} \tag{10}$$

into the mass equation (8):

$$\rho_t + \nabla\rho \cdot \vec{u} + \rho \nabla \cdot \vec{u} = 0 \tag{11}$$

$$\frac{\partial}{\partial t}\rho + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\rho \frac{\dot{a}_i(t)}{a_i(t)}(x_i + d_i) + \sum_{i=1}^{N} \frac{\rho \dot{a}_i(t)}{a_i(t)} = 0. \tag{12}$$

The general solutions for the semi-linear partial differential equation (12) are:

$$F\left(\prod_{i=1}^{N} a_i(t)\rho, \frac{x_1+d_1}{a_1(t)}, \frac{x_2+d_2}{a_2(t)}, ...., \frac{x_N+d_N}{a_N(t)}\right) = 0 \tag{13}$$

with an arbitrary $C^1$ function $F$ such that $\rho \geq 0$.

We take the explicit one as

$$\rho = \frac{f\left(\frac{x_1+d_1}{a_1(t)}, \frac{x_2+d_2}{a_2(t)}, ...., \frac{x_N+d_N}{a_N(t)}\right)}{\prod\limits_{i=1}^{N} a_i(t)}. \tag{14}$$

Therefore, the proof is completed. ■

The following proof is the checking our constructed functions (3) for the main result:

**Proof of Theorem 1.** We observe that the functions (3) satisfy the conditions in Lemma 3 for the mass equation $(2)_1$. Alternatively, readers can plug the functions (3) to balance the mass equation by the directional computation like in [4] and [7].

For the $i$-th momentum equation $(2)_2$ of the Euler and Navier-Stokes equations, we define the self-similar variable with elliptic symmetry (for not $a_1 = a_2 = .... = a_N$) and drift phenomena (for not all $d_i = 0$):

$$s = \sum_{k=1}^{N} \frac{(x_k + d_k)^2}{a_k(t)^2} \tag{15}$$

to have:

$$\rho \left[ \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} \right] + K \frac{\partial}{\partial x_i} \rho^{\gamma} + \mu \Delta u_i \tag{16}$$

$$= \rho \left[ \frac{\partial}{\partial t} \left( \frac{\dot{a}_i}{a_i} (x_i + d_i) \right) + \left( \frac{\dot{a}_i}{a_i} (x_i + d_i) \right) \frac{\partial}{\partial x_i} \left( \frac{\dot{a}_i}{a_i} (x_i + d_i) \right) \right] + K\gamma \rho^{\gamma - 1} \frac{\partial}{\partial x_i} \rho \tag{17}$$

$$= \rho \left\{ \left[ \left( \frac{\ddot{a}_i}{a_i} - \frac{(\dot{a}_i)^2}{(a_i)^2} \right) (x_i + d_i) + \frac{(\dot{a}_i)^2}{(a_i)^2} (x_i + d_i) \right] + K\gamma \rho^{\gamma - 2} \frac{\partial}{\partial x_i} \frac{f(s)}{\prod\limits_{k=1}^{N} a_k} \right\} \tag{18}$$

$$= \rho \left\{ \frac{\ddot{a}_i}{a_i} (x_i + d_i) + 2K\gamma \frac{f(s)^{\gamma-2}}{\left( \prod\limits_{k=1}^{N} a_k \right)^{\gamma - 2}} \frac{\dot{f}(s)}{\left( \prod\limits_{k=1}^{N} a_k \right)} \left( \frac{x_i + d_i}{a_i^2} \right) \right\} \tag{19}$$

$$= \frac{(x_i + d_i)\rho}{a_i^2} \left\{ \ddot{a}_i a_i + 2K\gamma \frac{f(s)^{\gamma-2} \dot{f}(s)}{\left( \prod\limits_{k=1}^{N} a_k \right)^{\gamma-1}} \right\} \tag{20}$$

$$= \frac{(x_i + d_i)\rho}{a_i^2 \left( \prod\limits_{k=1}^{N} a_k \right)^{\gamma-1}} \left\{ \xi + 2K\gamma f(s)^{\gamma-2} \dot{f}(s) \right\} \tag{21}$$

with the $N$-dimensional Emden dynamical system

$$\begin{cases} \ddot{a}_i(t) = \frac{\xi}{a_i(t) \left( \prod\limits_{k=1}^{N} a_k(t) \right)^{\gamma-1}} \text{ for } i = 1, 2, ..., N \\ a_i(0) = a_{i0} > 0, \ \dot{a}_i(0) = a_{i1} \end{cases} \tag{22}$$

with arbitrary constants $\xi$, $a_{i0}$ and $a_{i1}$.

Here, the local existence for the Emden dynamical system (22) can be guaranteed by the fixed point theorem. Then, we further require the first order ordinary differential equation:

$$\begin{cases} \frac{\xi}{2K\gamma} + f(s)^{\gamma-2} \dot{f}(s) = 0 \\ f(0) = \alpha \geq 0. \end{cases} \tag{23}$$

The above equation (23) can be solved exactly by

$$f(s)=\begin{cases}\alpha e^{-\frac{\xi}{2K}s} & \text{for } \gamma=1\\ \max\left(\left(-\frac{\xi(\gamma-1)}{2K\gamma}s+\alpha\right)^{\frac{1}{\gamma-1}},\ 0\right) & \text{for } \gamma>1.\end{cases} \tag{24}$$

Thus, the functions (3) are the solutions for the Euler and Navier-Stokes equations (2).

In particular, with $\gamma=1$, as the Emden dynamical system (22) becomes to be the conventional Emden equation:

$$\begin{cases}\ddot{a}_i(t)=\frac{\xi}{a_i(t)}\\ a_i(0)=a_{i0}>0,\ \dot{a}_i(0)=a_{i1},\end{cases} \tag{25}$$

we may use the energy method in classical mechanics (or readers may refer the lemma 7 in [6]) to show that (1a) for $\xi<0$, functions $a_i(t)$ blow up on a finite time;

(1b) for $\xi>0$, the functions $a_i(t)$ exist globally.

With $\gamma>1$,

(2a) for $\xi<0$ and some $a_{i1}<0$, by comparing the second order linear ordinary differential equations:

$$\begin{cases}\ddot{a}_i\leq 0\\ a_i(0)=a_{i0}>0,\ \dot{a}_i(0)=a_{i1},\end{cases} \tag{26}$$

we can show that the solutions (3) blow up on or before the finite time

$$T=\min(-a_{i0}/a_{i1}:a_{i1}<0,\ i=1,2,...,N); \tag{27}$$

(2b) for $\xi>0$ and all $a_{i1}\geq 0$, similarly, it is clear for that the solutions (3) exist globally.

The proof is completed. ■

# 3 Conclusion and Discussion

In this brief paper, the analytically self-similar solutions (3) with elliptic symmetry and drift phenomena for the compressible Euler and Navier-Stokes equations in $R^N$ $(N\geq 2)$ are constructed by the separation method. We reduce the Euler and Navier-Stokes equations (2) into the $1+N$ differential functional equations:

$$(f(s),a_i(t)\text{ for } i=1,2,....N). \tag{28}$$

In addition, by analyzing the Emden dynamical system (5):

$$\begin{cases}\ddot{a}_i(t)=\dfrac{\xi}{a_i(t)\left(\prod\limits_{k=1}^{N}a_k(t)\right)^{\gamma-1}},\ \text{for } i=1,2,....,N\\ a_i(0)=a_{i0}>0,\ \dot{a}_i(0)=a_{i1},\end{cases} \tag{29}$$

some blowup or global properties of the constructed solutions (3) can be shown easily.

However, for the $N$-dimensional ($N \geq 2$) Emden dynamical system (29) with arbitrary constants $\xi$, $\gamma$, $a_{i0}$ and $a_{i1}$, the all blowup sets, blowup times and asymptotic analysis of the solutions are not clear to be obtained. Computing simulation and rigorous mathematical proofs for the system (29) can be followed to understand the evolution of the constructed flows (3) for the Euler and Navier-Stokes equations in the future.

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